

A nerve lemma for gluing together incoherent discrete Morse functions

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September 7, 2012

Abstract

Two of the most useful tools in topological combinatorics are the nerve lemma and discrete Morse theory. In this note we introduce a theorem that interpolates between them and allows decompositions of complexes into non-contractible pieces as long as discrete Morse theory ensures that they behave well enough. The proof is based on diagrams of spaces, but that theory is not needed for the formulation or applications of the theorem.

1 Introduction

Two workhorses of topological combinatorics are the nerve lemma and discrete Morse theory. In this note we present a formula that interpolates between them, and allows you to glue together incoherent discrete Morse functions.

To state the main theorem we need one piece of notation. When recursively building the Morse complex by gluing in critical cells, the first critical cell considered is the *initial critical cell*. When there are several critical vertices there is a choice to be made, but the following theorem doesn't depend on that.

Theorem 1.1. *Let X be a regular CW complex with subcomplexes X_1, \dots, X_n satisfying $X = \cup_{i=1}^n X_i$, and let $P = \{\cap_{i \in I} X_i \mid I \subset \{1, \dots, n\}\}$ be a poset with $Y \leq Y'$ whenever $Y \supseteq Y'$. Fix a Morse matching on each $Y \in P$. If $\dim \sigma_Y > \dim \sigma_Z$ for all comparable subspaces $Y \supset Z$ in P with non-initial critical cells $\sigma_Y \in Y$ and $\sigma_Z \in Z$, then*

$$X \simeq \bigvee_{Y \in P} Y * \Delta(P_{<Y}).$$

Why does Theorem 1.1 interpolate between the nerve lemma and discrete Morse theory?

There are contractible but not collapsible regular CW complexes, but they only appear to verify that fact. In most applications of the nerve lemma all

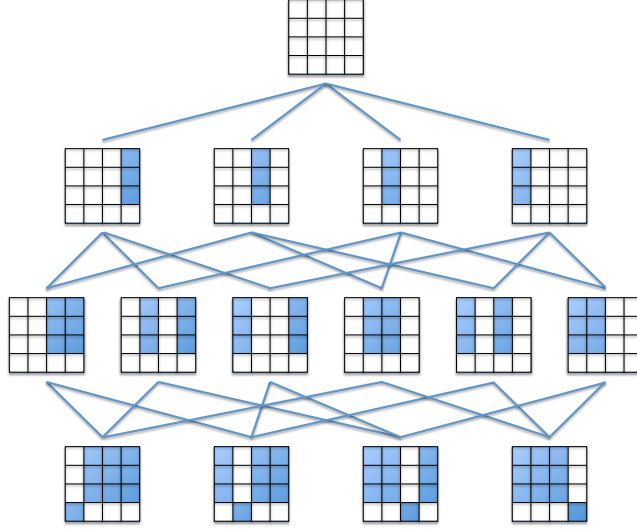


Figure 1: The intersection poset for calculating the homotopy type of a 4×4 chessboard complex.

contractible complexes are collapsible, and that is exactly Theorem 1.1 restricted to the case of Morse matchings with only one critical cell. Ordinary discrete Morse theory is the restriction to one subcomplex, the complex itself.

In the next section we give an example of how to apply the formula, and in the last section we prove it.



2 An example


The vertices of the 4×4 chessboard complex $\square\square$ are the 4×4 different places one could put a rook on the board, and the $4!$ facets correspond to the ways one can place four rooks without having them attacking each other. In general, the vertices of $n \times n$ chessboard complexes also correspond to bijections between n elements, and there is a huge theory on topological, combinatorial and representation theoretic results. See for example [6] for a survey.

Now we define four subcomplexes of $\square\square$ as the vertex induced subcomplexes defined by the shadowed positions in the bottom row of the poset in Figure 1. The topological situations for different intersections are as follows:

- ($Y = \square\square$) The complex Y is a star of a vertex, and a cone with that vertex as apex. Every cone has a complete Morse matching with only one critical cell, the initial one. Thus, $Y * \Delta(P_{<Y}) \simeq \cdot * \emptyset \simeq \cdot$.
- ($Y = \square\square$) The complex Y is an S^1 realized as cycle on six edges and six vertices. There is a Morse function with one non-initial critical cell of dimension one,

and one initial critical cell of dimension zero. The order complex $\Delta(P_{<Y})$ is two disjoint points, denoted S^0 . Thus, $Y * \Delta(P_{<Y}) = S^1 * S^0 \simeq S^2$.

- (Y=) The complex Y consists of three disjoint vertices, and every discrete Morse function makes them all critical. The poset $P_{<Y}$ is the face poset of the boundary of a triangle, and $\Delta(P_{<Y})$ is its barycentric subdivision. $Y * \Delta(P_{<Y}) = (\cdots) * S^1 \simeq S^2 \vee S^2$.
- (Y=) The complex Y is empty. The poset $P_{<Y}$ is the face poset of the boundary of a tetrahedron, and $\Delta(P_{<Y})$ is its barycentric subdivision. Thus, $Y * \Delta(P_{<Y}) = \emptyset * S^2 \simeq S^2$.

Compiling the facts, we get that  is homotopy equivalent to a wedge of 15 two-spheres. This result is well-known, but the computations were sometimes more cumbersome.

3 Proof of the formula

Before proving the formula, we point out some references to the methods used in it. As a general reference for topological combinatorics, Björner's survey [1] is a good start. The foundational paper on discrete Morse theory was by Forman [2], and for a survey on contemporary usage, see the book by Jonsson [4]. We build on the explicit proof of the main theorem of discrete Morse theory provided in Section 3 of [5]. For diagrams of spaces, the paper by Welker, Ziegler and Živaljević [7] is recommended. All of these tools were also used by Engström to find topological representations of matroids by homotopy colimits, and are briefly surveyed in [3].

Proof. (of Theorem 1.1) Construct an intersection diagram \mathcal{D} based on $X = \cup_{i=1}^n X_i$. This is a diagram on the poset P , but the language of diagrams of spaces was avoided in the theorem statement. Intersection diagrams are cofibrant, so $X = \text{colim } \mathcal{D} \simeq \text{hocolim } \mathcal{D}$.

For a space Y with a discrete Morse matching, there are many homotopy equivalences $f : Y \rightarrow Y^M$ to the Morse complex. We want to employ a particular type. Let $y = Y_1 \subset Y_2 \cdots \subset Y_m = Y$ be a filtration such that y is the initial critical vertex and in each step either two matched cells or a critical cell is added. This induces a filtration on the Morse complex $y^M = Y_1^M \subset Y_2^M \cdots \subset Y_m^M = Y^M$ and a sequence of homotopy equivalences $f_i : Y_i \rightarrow Y_i^M$ satisfying:

- (i) If Y_i extends Y_{i-1} by a critical cell, then f_i extends f_{i-1} by an homeomorphism from the open cell $Y_i \setminus Y_{i-1}$ to the open cell $Y_i^M \setminus Y_{i-1}^M$;
- (ii) If Y_i extends Y_{i-1} by a matching the open cells $\partial\tau \supset \sigma$, then for some homotopy equivalence $\pi : \bar{\tau} \rightarrow \partial\tau \setminus \sigma$ that is constant on its image, f_{i-1} extends by setting $f_i(x) = f_{i-1}(\pi(x))$ for $x \in \bar{\tau} \setminus (\partial\tau \setminus \sigma)$.

A crucial property for the homotopy equivalence $f : Y \rightarrow Y^M$ is that if $x \in Y$ is in the interior of a d -dimensional cell and $f(x)$ is in the interior of a d^M -dimensional cell, then $d \geq d^M$.

For each space Y in \mathcal{D} a discrete Morse function is given, together with a Morse complex Y^M , a homotopy equivalence $f_Y : Y \rightarrow Y^M$, and an initial critical cell y with its image $y^M = f_Y(y)$. For any two comparable spaces $Y \supset Z$ in \mathcal{D} , the diagram

$$\begin{array}{ccc} Z & \xrightarrow{f_Z} & Z^M \\ \downarrow & & \downarrow z \mapsto y^M \\ Y & \xrightarrow{f_Y} & Y^M \end{array}$$

commutes, since the dimension of any non-initial critical cell of Z is less than the dimension of any non-initial critical cell of Y . Construct a diagram \mathcal{D}^M on the same poset as \mathcal{D} but replace each complex by its Morse complex, and each map by a constant map to the image of the initial critical cell in the corresponding Morse complex. By commutativity of the preceding diagram, $\text{hocolim } \mathcal{D} \simeq \text{hocolim } \mathcal{D}^M$. Every map in \mathcal{D}^M is constant, so $\text{hocolim } \mathcal{D}^M \simeq \vee_{Y \in P} Y * \Delta(P_{<Y})$. \square

Acknowledgments.

This note was prepared for a summer school on discrete Morse theory and commutative algebra, organized by Bruno Benedetti and the author, July 18 – August 2, 2012, at Institut Mittag-Leffler. We thank the institute for being the perfect hosts for our school.

References

- [1] Anders Björner. Topological methods. *Handbook of combinatorics*, Vol. 1, 2, 1819–1872, Elsevier, Amsterdam, 1995.
- [2] Robin Forman. Morse theory for cell complexes. *Adv. Math.* **134** (1998), no. 1, 90–145.
- [3] Alexander Engström. Topological representation of matroids from diagrams of spaces, [arXiv:1002.3441](#), 18 pp.
- [4] Jakob Jonsson. *Simplicial complexes of graphs*. Lecture Notes in Mathematics, 1928. Springer-Verlag, Berlin, 2008. xiv+378 pp.
- [5] Dmitry Kozlov. Collapsibility of $\Delta(\Pi_n)/S_n$ and some related CW complexes. *Proc. Amer. Math. Soc.* **128** (2000), no. 8, 2253–2259.
- [6] John Shareshian and Michelle Wachs. Torsion in the matching complex and chessboard complex. *Adv. Math.* **212** (2007), no. 2, 525–570.
- [7] Volkmar Welker, Günter Ziegler and Rade Živaljević. Homotopy colimits – comparison lemmas for combinatorial applications. *J. Reine Angew. Math.* **509** (1999), 117–149.